MATH 2020A
Fall 2020

Lecture 22

- Divergence theorem

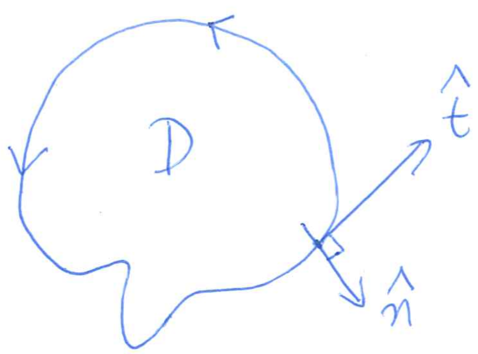
Recall that there are 2 forms of Green's theorem.

Tangent Form:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C P\,dx + Q\,dy$$

$$= \oint_C \vec{F} \cdot d\vec{r}$$

$$= \oint_C \vec{F} \cdot \hat{t}\, ds$$



and

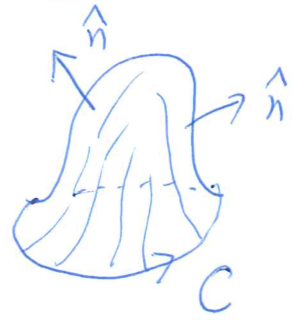
Normal form

$$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \oint -Q\,dx + P\,dy$$

$$= \oint_C \vec{F} \cdot \hat{n}\, ds$$

Tangent Form has been extended $D \to S \subset \mathbb{R}^3$, $C \to C \subset \mathbb{R}^3$:

<u>Stokes' Thm</u> $\displaystyle\iint_S \nabla \times \vec{F} \cdot \hat{n}\, d\sigma = \oint_C P\,dx + Q\,dy + R\,dz.$



Now normal form can be extended in another way.

Theorem (Divergence Thm, Gauss thm) Let $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ be a C^1-v.f. in some open set $G \subseteq \mathbb{R}^3$. Let S be a closed surface in G which bounds a region $\Omega \subseteq G$. Then

$$\iiint_{\Omega} \nabla\cdot\vec{F}\, dV = \oiint_{S} \vec{F}\cdot\hat{n}\, d\sigma, \text{ where}$$

$\hat{n}$ is the outward unit normal.

Here $\nabla\cdot\vec{F}$ is the divergence of $\vec{F}$:

$$\nabla\cdot\vec{F},\ \operatorname{div}\vec{F} \overset{\text{def}}{=} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Pf. Only consider the special case that Ω can be expressed as

$$\{(x,y,z) : (x,y)\in D,\ f_1(x,y) \le z \le f_2(x,y)\}, \text{ or}$$

$$\{(x,y,z) : (y,z)\in D,\ g_1(y,z) \le x \le g_2(y,z)\}, \text{ or}$$

$$\{(x,y,z) : (x,z)\in D,\ h_1(x,z) \le z \le h_2(x,z)\}.$$

In the first case, we'll show

$$\iiint_{\Omega} \frac{\partial R}{\partial z}\, dV = \oiint_{S} R\hat{k}\cdot\hat{n}\, d\sigma. \quad (1)$$

In the second case,

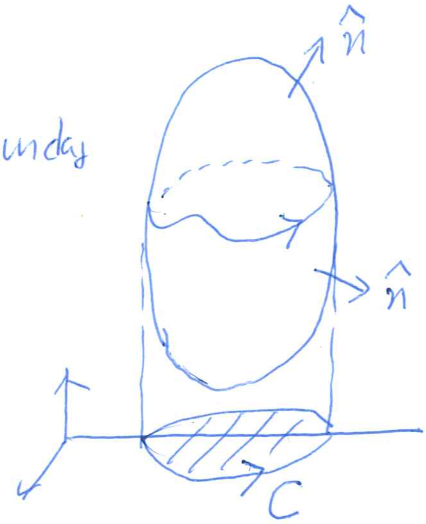
$$\iiint_{\Omega} \frac{\partial P}{\partial x}\, dV = \oiint_{S} P\hat{i}\cdot\hat{n}\, d\sigma.$$

In the third case,

$$\iiint_{\Omega} \frac{\partial Q}{\partial y}\, dV = \oiint_{S} Q \hat{j}\cdot\hat{n}\, d\sigma.$$

By adding up these third formulas, we obtain the divergence thm.

We'll prove the 1st case only (the other two cases can be treated similarly.)



Assume that f_1 meets f_2 at the boundary so that

$S = S_1 \cup S_2$, where

$S_2 = \{(x, y, f_2(x,y)) : (x,y) \in D\}$

$S_1 = \{(x, y, f_1(x,y)) : (x,y) \in D\}$

RHS of ① $\displaystyle \oiint_S R\hat{k}\cdot\hat{n}\, d\sigma = \iint_{S_1} R\hat{k}\cdot\hat{n}\, d\sigma + \iint_{S_2} R\hat{k}\cdot\hat{n}\, d\sigma.$

Since $\displaystyle \hat{n} = \frac{(-f_{2x}, -f_{2y}, 1)}{\sqrt{1+|\nabla f_2|^2}}$ on S_2,

$$\iint_{S_1} R\hat{k}\cdot\hat{n}\, d\sigma = \iint_{S_2} R\frac{1}{\sqrt{1+|\nabla f_2|^2}}\, d\sigma = \iint_{D} R(x,y,f_2(x,y))\, dA(x,y).$$

Since $\displaystyle \hat{n} = \frac{(f_{1x}, f_{2x}, -1)}{\sqrt{1+|\nabla f_1|^2}}$ on S_1, ($\hat{n}$ points downward on S_1)

$$\iint_{S_1} R\hat{k}\cdot\hat{n}\, d\sigma = -\iint_{D} R(x,y,f_1(x,y))\, dA(x,y).$$

$\therefore$ RHS of ①:

$$\iint_D R(x,y,f_2(x,y)) - R(x,y,f_1(x,y))\, dA(x,y) \quad \text{— ②}$$

On the other hand,

$$LHS = \iiint_\Omega \frac{\partial R}{\partial z}\, dV \qquad (\text{Fubini's thm})$$

$$= \iint_D \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial R}{\partial z}\, dz\, dA$$

$$= \iint_D \big(R(x,y,f_2(x,y)) - R(x,y,f_1(x,y))\big)\, dA(x,y)$$

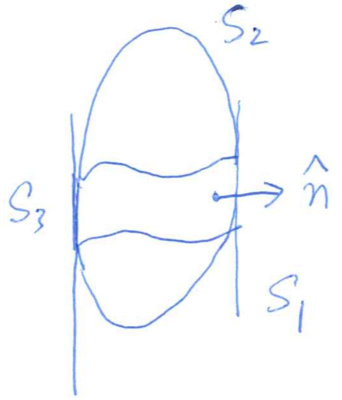
$$= ②.$$

<u>Remark</u> In case $f_1(x,y) < f_2(x,y)$ on the boundary,

$\hat{n}\cdot\hat{k} = 0$ on S_3

$$\iint_{S_3} R(x,y,z)\,\hat{n}\cdot\hat{k} = 0,$$

it has no contribution.



$S = S_1 \cup S_2 \cup S_3$

As like the Green's thm, when Ω is bdd by several closed surfaces $S_1, \ldots, S_n$,

$$\iiint_{\Omega} \nabla \cdot \vec{F}\, dV = \sum_{j=1}^{n} \oiint_{S_j} \vec{F} \cdot \hat{n}\, d\sigma .$$

$\boxed{\text{e.g. }1}$. Let $\vec{F} = xy\,\hat{i} + yz\,\hat{j} + xz\,\hat{k}$. Find its outward flux across the cube at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Use
$$\oiint_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint_C \operatorname{div} \vec{F}\, dV .$$

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = y + z + x$$

$$\therefore \oiint_S \vec{F} \cdot \hat{n}\, d\sigma = \int_0^1 \int_0^1 \int_0^1 (y+z+x)\, dx\, dy\, dz$$

$$= 3 \int_0^1 \int_0^1 \int_0^1 x\, dx\, dy\, dz \qquad \text{(by symmetry)}$$

$$= 3/2 \ \#$$

$\boxed{\text{e.g. }2}$ The electric field generated by a point charge q at the origin is denoted by $\vec{E}$,

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{1}{\rho^3} (x, y, z), \qquad \varepsilon_0 > 0 \text{ constant}.$$

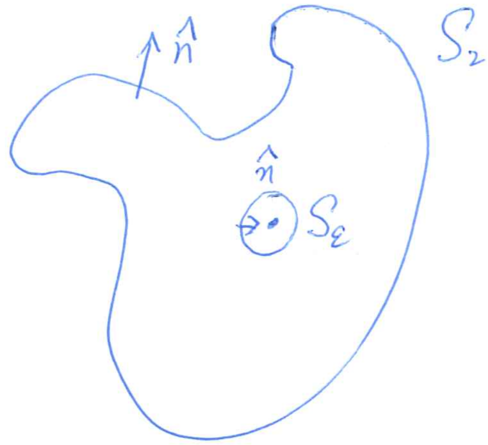
Prove Gauss' theorem: The outward flux across any closed surface enclosing the origin is always equal to q/ε_0.

Well, $$\nabla\cdot\vec{E} = \frac{q}{4\pi\varepsilon_0}\,\nabla\cdot\frac{1}{\rho^3}(x,y,z)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{y^2+z^2-2x^2}{\rho^5} + \frac{x^2+z^2-2y^2}{\rho^5} + \frac{y^2+x^2-2z^2}{\rho^5}\right)$$

$$= 0.$$

Let S be a simple, closed surface enclosing $(0,0,0)$ and let S_ε be the sphere of radius ε at $(0,0,0)$, ε so small that S_ε is contained by S. By div. thm,

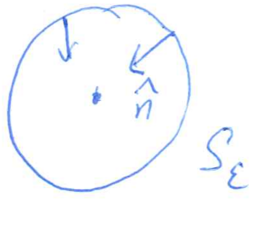


$$\iint_S \vec{E}\cdot\hat{n}\,ds + \iint_{S_\varepsilon} \vec{E}\cdot\hat{n}\,ds$$

$$= \iiint_{\Omega_\varepsilon} \nabla\cdot\vec{E}\,dV = 0,$$

when Ω_ε is the region bdd between S and S_ε. Since $(0,0,0)$ lies outside Ω_ε, $\nabla\cdot\vec{E}=0$ in Ω_ε. Noting that here $\iint_{S_\varepsilon}\vec{E}\cdot\hat{n}\,ds$ is the flux pointing out of Ω_ε, so

$-\iint_{S_\varepsilon}\vec{E}\cdot\hat{n}\,ds$ is the flux pointing outward from S_ε



$$= \frac{q}{4\pi\varepsilon_0}\iint_{S_\varepsilon} \frac{1}{\varepsilon^3}(x,y,z)\cdot\frac{(x,y,z)}{\varepsilon}\,d\sigma$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{\varepsilon^4}\iint_{S_\varepsilon}(x^2+y^2+z^2)\,d\sigma$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{\varepsilon^4}\times\varepsilon^2\iint_{S_\varepsilon}d\sigma = \frac{q}{4\pi\varepsilon_0}\frac{1}{\varepsilon^4}\varepsilon^2\times 4\pi\varepsilon^2$$

$$= \frac{q}{\varepsilon_0}.$$ #